Novel fast algebraic direct method solver for linear symmetric systems

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Direct methods of linear algebraic systems solving methods give an exact solution in a definite number of operations. These methods are stable. However, they consume a large memory space for storage. Mesh and graph renumbering methods can accelerate accurately these methods and reducing memory storage. The fastest of these direct methods is the Cholesky method for envelope storage. It has the drawback of being applied only for symmetrical positive definite matrix systems, which represents an obstacle for coupled circuits (electric - magnetic). In this paper, we present a new solution technique faster and more general than the Cholesky method. The methods allow a significant reduction of CPU time consuming which is very suitable in time stepping finite element large problems.

***Index Terms*— algebraic systems, coupled circuits, direct solvers, finite elements method, sparse matrices**

I. Introduction

FINITE element method is today one of the most popular numerical methods for solving partial differential equations. Since the seventies, it is applied to compute electromagnetic devices behavior. It consists of transforming a continuous differential equation to an algebraic system of equations. The obtained stiffness matrix is generally an extremely sparse symmetric matrix [1]. Direct and iterative methods can be used for solving such systems. Direct methods provide an exact solution using a finite number of operations without difficulties of convergence which are encountered with iterative methods. The most widely used direct methods are variants of Gaussian elimination and involve the explicit factorization of the system matrix A (or, more usually, a permutation of A) into a product of lower and upper triangular matrices. Moreover, finding and computing a good preconditioner for use with an iterative method can be computationally more expensive than using a direct method [2].

These properties are advantageous when using time stepping methods with direct methods. We can remove cumulative errors due to the iterative process. However, direct methods require large memory resources and consume a large CPU time. Some renumbering methods are developed for reducing bandwidth and/or envelope. Storage is then reduced, and number of operations is also reduced [1, 3].

Cholesky and frontal methods (based on Gauss elimination method) are preferred for solving these obtained algebraic systems. For coupled circuits, the stiffness matrix is completed, and the system is not definite positive [4].

In this paper, the authors propose a novel and original method to solve a symmetric algebraic system, in its square form, faster than gauss and Cholesky in its first form.

II. Principle of the proposed method

Let A be an N by N symmetric matrix, with a_{ij} entries. To solve the following algebraic system:

$$A\,x = b \tag{1}$$

Some authors propose to transform the system in an equivalent system with a triangular matrix (Like Gauss), a product of two triangular matrices (Cholesky…) or diagonal matrix (Jordan…)…etc. We can note that the matrix A is a definite positive matrix. The proposed method transforms A in triangular matrix with an unity diagonal.

$$A'x = b' \tag{2}$$

If we consider that the matrix A' is a sum of three matrices: a strictly upper (U), a strictly lower (L) and a diagonal matrix (D), we wrote:

$$A' = (L) + (D) + (U) \tag{3}$$

The presented method considers:

$$(L) = 0 \tag{4}$$

$$(D) = (I) \text{ matrix unity} \tag{5}$$

Like many methods, the presented method has two steps:

- Factorization
- Resolution

Factorization needs about the same number of operations as Cholesky factorization.

For the resolution, equation (2) is written as:

$$x = -(U)\,x + b' \tag{6}$$

This consists in a product upper matrix- vector and a sum with a vector.

We consider, for example, this system:

$$\begin{pmatrix} -4 & 1 & 1 & 1 \\ 1 & -4 & 2 & 2 \\ 1 & 2 & -4 & 1 \\ 1 & 2 & 1 & -5 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \\ -4 \\ 0 \end{pmatrix} \tag{7}$$

It is transformed to the following form:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 & 0.25 & 0.25 & 0.25 \\ 0 & 0 & 0.6 & 0.6 \\ 0 & 0 & 0 & 1.083 \\ 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} + \begin{pmatrix} 0 \\ -0.8 \\ 0.917 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 2 \\ 1 \end{pmatrix} \tag{8}$$

If we change the second member (b) the (U) matrix does not change.

$$\begin{pmatrix} -4 & 1 & 1 & 1 \\ 1 & -4 & 2 & 2 \\ 1 & 2 & -4 & 1 \\ 1 & 2 & 1 & -5 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ -3 \\ 2 \\ 1 \end{pmatrix} \quad (9)$$

and its solution is:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 & 0.25 & 0.25 & 0.25 \\ 0 & 0 & 0.6 & 0.6 \\ 0 & 0 & 0 & 1.083 \\ 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} + \begin{pmatrix} 0 \\ 0.8 \\ -0.083 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 1 \\ 1 \end{pmatrix} \quad (10)$$

This means that in a PDE problems (electromagnetic, thermal, or coupled ...) with time varying sources (second term *b*), the factorization of the stiffness matrix is performed once unlike Gaussian elimination methods. This is suitable for CPU time saving for large systems in Time stepping Finite element method.

This method has three excellent advantages:

- It is applicable for general symmetric systems;
- It has the same number of operations for factorization as Cholesky' method;
- For a repetitive computations (for example, in a stepping method with fixed mesh), stiffness matrix *A* is determined only at first step.

III. APPLICATION

For application, we want to compare performances with Gauss and Cholesky methods. We consider a heat flow problem in Totally Enclosed Fan Cooled ''TEFC'' induction machine (2,2 kW). Equation to solve is :

$$\lambda\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right) + P = 0 \quad (10)$$

where λ is the thermal diffusivity an *P* is the heat source.

Only a part of structure is studied (Fig.1).

IV. NUMERICAL RESULTS

We consider the stiffness matrix is stored in square matrix, and the Poisson's equation is solved with three methods: The proposed method, and, for comparison, the elimination Gauss method and Cholesky method.

Table. I, shows the total CPU time with the 3 methods for different values of mesh node number. It can be seen that the proposed method is almost 20% faster than Cholesky method.

In order to accelerate the resolution, a renumbering method that reduces the bandwidth is applied [3]. The matrix is also stored in square form, but algorithms are modified to take into account the reduced bandwidth. Results are given in Tab. II.

In this case, It can be seen that the proposed method has the same speed as Gauss method. The two methods are faster than Cholesky Method. However for with Gauss elimination method the matrix is transformed at each time step which is not the case of the proposed method.

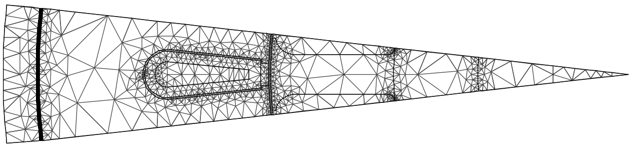

Fig. 1.Studied Induction machine in heat flow problem with finite elements method.

TABLE I
CPU TIMES IN SECONDS FOR DIFFERENT SYSTEM SIZES FOR THE 3 METHODS

N (nodes)	Gauss	Cholesky	Proposed Method
991	11.69	6.94	5.64
1160	20.14	11.79	9.65
2119	132.1	78.04	63.34
2618	251.1	151.1	120.4

TABLE II
CPU TIME IN SECONDS FOR DIFFERENT SYSTEM SIZES FOR THE 3 METHODS WHILE CONSIDERING MINIMIZING MAXIMUM BANDWIDTH

Nodes	Bandwidth	Gauss	Proposed Method
1160	71	0.19	0.18
2119	115	0.88	0.88
2618	119	1.18	1.19
3655	127	1.94	1.93

For a complete comparison, we need to apply a profile reduction and improve algorithms for this storage.

V.CONCLUSION

The authors have presented a novel and original method to solve the algebraic linear system. This method presents many advantages. It is, in its form, faster than Cholesky method.

It is applied for all symmetric systems and not limited for definite positive systems, like Cholesky. It is suitable for coupled FEM-circuits problems

The matrix *A* is calculated only at the first step when *b* varies. The proposed method is suitable for time stepping methods. The second member of equation is simple to calculate from the modified matrix *A'*

In bandwidth reduction, the proposed method is also faster than Cholesky method. It has similar performances than Gauss method. We can see then that the proposed method is very suitable for coupled circuits and step by step finite elements analysis

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